Chapter 1 Review of Basic Mathematics

Numbers

Natural numbers are the basic and familiar numbers, also called positive integers: 1,2,3,4,… Together with the negative integers, -1,-2,-3, …, they make up the *integers.* These integers can be even (2,4,6,…) or odd (1,3,7,...).

Rational numbers are the numbers in the form $\frac{d}{e}$, in which d and e are both integers, and $e \neq 0$. For $e = 0$ the fraction cannot be defined for any real number d. Note that any integer n is a rational number because $n = \frac{n}{1}$.

The most common system of writing numbers is the *decimal system*. It is a system in which 10 is the base number and every natural number can be written using only digits (1,2,3,4,5,6,7,8,9,0). It follows that each combination of digits can be written as a sum of powers of 10. For example:

$$1984.14 = 1 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 10^0 + \frac{1}{10^1} + \frac{4}{10^2}$$

Rational numbers can be *finite decimal fractions* or *infinite decimal fractions.* The latter is the case when the rational number cannot be written by using a finite number of decimal places. For example, $\frac{100}{3} = 33.333 \ldots$

Real numbers are arbitrary infinite decimal fractions, of the form $x = \pm m.\alpha_1\alpha_2\alpha_3 \ldots$.
In the case of rational numbers, the decimal fraction will always be recurring or periodic, that is, there is repetition in the decimal expansion ($\frac{11}{70} = 0.157142857142 85 \ldots$). If the decimal fraction is not periodic, then these numbers are called *irrational numbers.*

Powers

A 100 times multiplication of w can be written as w^{100}. W is called the base and 100 is the exponent. In general terms we can say that the expression $w^n = w \cdot w \cdot w \cdot w \cdot \ldots$ is called the n-th power of w.

The properties of powers are the following:

1. $w^0 = 1$
2. $w^a \times w^b = w^{a+b}$
3. $\frac{w^a}{w^b} = w^{a-b}$
4. $\frac{1}{w^a} = w^{-a}$
5. $(w^a)^b = w^{ab}$
6. $\left(\frac{w}{z}\right)^a = \frac{w^a}{z^a}$
7. $(wz)^a = w^a z^a$

Using algebra we can find that in general: $(w+z)^a \neq w^a + z^a$.

To give an example of a practical application, exponents are used to compound interest:

$$A = P\left[1 \pm \frac{r}{100}\right]^t$$

A is the total new amount, P is the initial amount, r is the rate of change, or the interest rate in percentages per year and t is time in years. It is an addition when the rate, r, is increasing, and a deduction when the rate in decreasing. 1+r100 is called the growth factor for a growth or decline of r%.

A numerical example is the following: Williams wins a price of 500 and he deposits this amount on his bank account that pays 6% interest per year. After 5 years he will have:

$$A = 500\left[1 + \frac{6}{100}\right]^5 = 500[1.06]^5 \cong €669.11$$

Algebra rules

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. $a + 0 = a$
4. $a + (-a) = 0$
5. $ab = ba$
6. $(ab)c = a(bc)$
7. $1 \cdot a = a$
8. $a.a^{-1} = 1, a \neq 0$
9. $(-a).b = a.(-b) = -a.b$
10. $(-a).(-b) = a.b$
11. $a(b+c) = a.b + a.c$
12. $(a + b)c = ac + bc$
13. $(a+b)^2 = a^2 + 2ab + b^2$
14. $(a-b)^2 = a^2 - 2ab + b^2$
15. $(a+b)(a-b) = a^2 - b^2$, also called the *difference-of-squares formula*
16. $(a)^{W/z} = \sqrt[z]{a^W}$

Fractions

A fraction is called proper when the numerator is smaller than the denominator, $\frac{3}{9}$, and a fraction is called improper when the numerator is larger than the denominator, $\frac{9}{3}$.
We list the most essential properties of fractions below:

1. $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$
2. $\frac{-a}{-b} = \frac{a}{b}$
3. $-\frac{a}{b} = \frac{-a}{b}$
4. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
5. $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$
6. $a + \frac{b}{c} = \frac{a \cdot c + b}{c}$
7. $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$
8. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
9. $\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$

The first rule is very important because it can be used to simplify fractions by factoring the numerator and the denominator, or in other words, by cancelling common factors.

Fractional powers
In economics fractions in powers are common. Fractional powers have some important algebraic consequences:

1. $a^{1/2} = \sqrt{a}$, valid only if $a \geq 0$
2. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
3. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
4. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
5. $a^{\frac{1}{n}} = \sqrt[n]{a}$
6. $a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^{p} = (a^{p})^{\frac{1}{q}}$

Inequalities

If a number *a* is strictly greater than a number *b*, we write $a > b$. If *a* is greater than or equal to *b,* we write $a \geq b$. If a number *a* is strictly smaller than a number *b*, we write $a < b$. And we write $a \leq b$ when *a* is smaller than or equal to b. There are six fundamental properties related to inequalities:

1. $a > 0$ and $b > 0$ imply $a + b > 0$ and $a \cdot b > 0$
2. If $a > b$, then $a + c > b + c$
3. If $a > b$ and $b > c$, then $a > c$
4. If $a > b$ and $c > 0$, then $ac > bc$
5. If $a > b$ and $c < 0$, then $ac < bc$
6. If $a > b$ and $c > d$, then $a + c > b + d$

These properties remain valid when each $>$ is replaced by a $\geq$.

When the two sides of an inequality are multiplied by a positive number, the direction of the inequality is preserved. But when the two sides are multiplied by a negative number, the direction of the inequality is reversed. For example: when the inequality 30b > 20c is multiplied by -2, the inequality becomes -60b < -40c.

An example of how to solve the inequality $\frac{6a-2}{-5} \leq 10$

$=> multiplying\ both\ sides\ by -5, \frac{6a-2}{-5} \times -5 \geq 10 \times -5,$

$=> 6a - 2 \geq -50,$

$=> 6a \geq -50 + 2$

$=> a \geq \frac{-48}{6} \quad => \quad a \geq -8$

Sign Diagrams

Sometimes it can be useful to use a sign diagram to show all the possible values for an inequality. See pages 25, 26 and 27 for illustrations of such diagrams.

We use the following inequality to illustrate the sign diagram: $\frac{(a-6)}{a-3} > 2 - a$.

1. First bring all everything to one side of the inequality: $\frac{(a-6)}{a-3} - 2 + a > 0$
2. Then, make sure you have a common denominator and solve the numerator: $\frac{a^2-4a}{a-3} > 0$
3. The result can be used to make a sign diagram: $\frac{a(a-4)}{a-3} > 0$

Now for every part of the inequality there will be a horizontal number line, showing for which numbers the inequality holds and for which numbers it does not hold, hence in this case for which number the left-hand side is indeed positive. In this case we need to consider three parts and the whole: a, $a-4$, $a-3$, and $\frac{a(a-4)}{a-3}$.

Intervals

There are four different types of intervals. An interval is a set of numbers that lies between two points on a line. There are four types of *bounded* intervals:

(a, b)	Open interval	$a < x < b$
$[a, b]$	Closed interval	$a \leq x \leq b$
$(a, b]$	Half-open interval	$a < x \leq b$
$[a, b)$	Half-closed interval	$a \leq x < b$

Apart from these bounded intervals, an interval can also be unbounded when there is no upper limit. This happens in the case of infinity, for example in the case of the following interval: $[a, \infty)$. An interval going to infinity can never be completely closed.

Absolute value

The absolute value of a number *a* is the distance between the real number and zero on a number line. Is *a* positive then the absolute value is the number a itself. Is *a* negative, then the absolute value is actually –*a*. The denotation is as follows:

$$|a| = \begin{cases} a \; if \; a \geq 0 \\ -a \; if \; a < 0 \end{cases}$$

The distance between a and b on the number line = $|a-b| = |b-a|$

And $|x| < a$ means that $-a < x < a$. The $<$ can be replaced by $\leq$.