Chapter 8: Options

An *European* call option is the right to buy the underlying asset at the strike price, at a preset point in time. An *American* call option is the right to buy the underlying asset at any time on or before the expiration date of the option. The European option is easier to value than the American option.

Four attributes illustrate an option:

1. An underlying risky asset that determines the option's value at some future date
2. A strike price
3. A date before which the option cannot be exercised
4. An expiration date beyond which the option can no longer be exercised

Attribute 3 and 4 are identical for a European option.

Future cash flows are never positive when writing an option. To compensate the option writer for these future adverse consequences, the option buyer pays money to the writer to acquire the option.

The *put-call parity formula* links the prices of European calls to the prices of European puts. If no dividends are paid prior to expiration, then, with the assumption no arbitrage:

$$c_0 - p_0 = S_0 - PV(K)$$

A European call sells for more than a European put option when the option is at-the-money and has a strike price equal to the current stock price.

American options are only worth more than options if the right of premature exercise has value. But because a stock that pays no dividend European s before expiration generates a cash of $S_0 - K$ when it is exercised, while it gives a cash c_0 when selling the option. Clearly, it never pays to exercise an American option before the expiration date.

With an option or even with a forward contract on a non dividend-paying security, paying for the security at the latest date possible is logically. With an option, you have further incentive to wait: if the security later goes down in value, you can choose not to acquire the security by not exercising the option and, if it goes up, you can exercise the option and acquire the security.

With a put option, the holder receives rather than pays out cash upon the exercise, and the earlier the receipt of cash, the better. An investor trades off the interest earned from receiving cash early against the value gained from waiting to see how things will turn out.

The no-arbitrage value of American and European call options are the same when the underlying stock pays no dividends before the expiration date. We make the assumption here that the options have the same elements.

The difference between the no-arbitrage values of a European call and a European put with the same elements is the current price of the stock less the sum of the present value of the strike price and the present value of all dividends to expiration.

Equity can be thought of as a call option on the assets of the firm. The reason for this is the limited liability of the equity holders.

The more debt a firm has, the less in the money is the implicit option in equity. Because stock is an option on the organizations assets, a call option on the stock is actually an option on an option, which we name a *compound option*.

Portfolio insurance is an investment which is based on options and when added to the existing investments of a pension fund or mutual fund, it protects the fund's value at a target horizon date against drastic losses.

When portfolios exists of:

- A call option
- Riskless zero-coupon bonds

The portfolio's value at the date the options expire would never fall below the value of the riskless bonds, the floor amount, at that date. With this, the portfolio is insured.

The present value of the two components of an insured portfolio is $c_0 + PV(F)$

Where *PV(F)* is the floor amount and discounted at the risk-free rate.

The extended put-call parity formula, $c_0 - p_0 = S_0 - PV(K) - PV(div)$,

implies that the present value of the desired insured portfolio is:

$$c_0 + PV(F) = S_0 + p_0 - [PV(div) + PV(K) - PV(F)]$$

Where S_0 = the current value of the uninsured stock portfolio

p_0 = the cost of a put with a strike price of *K*

PV(div) = the present value of the uninsured stock portfolio's dividends

Valuing European options with the risk-neutral valuation method is only an issue of employing the risk-neutral probabilities to the expiration date values of the option and discounting the risk-neutral weighted average at the risk-free rate.

Suppose that the one-period risk-free rate is constant, and that the ratio of price in the following period to price in this period is always *u* or *d*. The hypothetical probabilities that would exist in a risk-neutral world must make the expected return on the stock equal the risk-free rate. The risk-neutral probabilities for the up and down moves that do this, π and $1-\pi$, respectively, satisfy:

$$\pi = \frac{1 + r_f - d}{u - d} \qquad \text{and } 1 - \pi = \frac{u - 1 - r_f}{u - d} \qquad \text{where}$$

u = ratio of next period's stock price to this period's price if the up state occurs

d = ratio of next period's stock price to this period's price if the down state occurs

The no arbitrage call value:

$$c_0 = \frac{\pi \max[uS_0 - K, 0] + (1-\pi)\max[dS_0 - K, 0]}{1 + r_f}$$

With *N* periods to the end, the risk-neutral expected value of the expiration value of the call, discounted at the risk-free rate, also brings forth the present no-arbitrage value of the call. There is only one pathway for *N* up moves, and the risk-neutral probability of coming there is π^N.

The worth of the option with a strike price of *K* at this point is $\max[0, u^N S_0 - K]$.

For N-1 up moves, the value of the option is $\max[0, u^{N-1} d S_0 - K]$, which multiplies the risk-neutral probability of $\pi^{N-1}(1-\pi)$.

In general, for *j* up moves, $j = 0,\ldots.,N$, each path has a risk-neutral probability of $\pi^j (1-\pi)^{N-j}$,

and there are $\frac{N!}{j!(N-j)!}$ such pathways to the associated value of max[0, $u^j d^{N-j} S_0 - K]$.

Therefore, the expected future value of a European call option, where the expectation uses the risk-neutral probabilities to weight the outcome, is $\sum_{j=0}^{N} \frac{N!}{j!(N-j)!} \pi^j (1-\pi)^{N-j} \max[0, u^j d^{N-j} S_0 - K]$

Where r_f = risk-free rate

π = risk-neutral probability of an up move

u = ratio of the stock price to the prior stock price given that the up state has occurred over a binomial step

d = ratio of the stock price to the prior stock price given that the down state has occurred over a binomial step

The procedure for modelling American option values with the binomial approach is alike to that for European options. Always work towards the backfrom the right-hand side of the tree diagram. At each node, look at the two future values of the option and use risk-neutral discounting to determine the value of the option at that node. This value, though, is only the value of the option at that node provided that the investor holds on to it for more than one period. If the investor exercises the option at the node, and the underlying asset is worth S at the node, then the value gained is S – K for a call and K – S for a put. Working backward at each node, make a comparison of (1) the value from early exercise of the option with (2) the value of waiting one more period and achieving one of two values. The value to be placed at that decision node is the larger of (1) and (2).

The cum-dividend value of a stock is the value previous to the expiration date. The ex-dividend value is the price of the stock after the expiration date.

Discrete models only take into account a limited amount of future results for the stock price and only a limited amount of points in time. *Continuous-time models* permit an never-ending number of stock price outcomes and they can depict the distribution of stock and option prices at any point in time.

The black-scholes formula

If a stock that pays no dividends before expiration of an option has a return that is log-normally distributed, can be continuously traded in frictionless markets, and has a constant variance, then, for a constant risk-free rate, the value of a European call option on that stock with a strike price of *K* and *T* years to expiration is given by

$$c_0 = S_0 N(d_1) - PV(K)N(d_1 - \sigma\sqrt{T}) \qquad \text{where} \quad d_1 = \frac{\ln(S_0 / PV(K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

The Greek letter σ is the annualized standard deviation of the natural logarithm of the stock return, l*n*( ) represents the natural logarithm, and *N(z)* is the probability that a normally distributed variable with a mean of zero and variance of 1 is less than *z*.

Delta is the change in the value of the derivative security with respect to movements in the stock price, holding everything else constant.

The delta of the option is the derivative of the option's price with respect to the stock price, $\delta c_0 / \delta S_0$ for a call, $\delta p_0 / \delta S_0$ for a put.

The derivative of the right-hand side of the Black-Scholes formula, with respect to S_0, the delta of the call option, is $N(d_1)$. Delta can be viewed as the number of shares of stock needed in the tracking portfolio. *x* is the number of shares in the tracking portfolio. For a given change in S_0, dS_0, then the change in the tracking portfolio is: xdS_0. Unless *x* equals $\delta c_0 / \delta S_0$ for a call or $\delta p_0 / \delta S_0$ for a put, xdS_0 will not be the same as the change in the call or put value, and the tracking of the option payoff with a portfolio of the underlying stock and a risk-free bond will not be perfect.

In the design of arbitrage, the ratio for the underlying asset position must be negative of the partial derivative of the theoretical option price with respect to the price of the underlying security. This partial derivative is $N(d_1)$, the option's delta. The first part of the Black-Scholes formula, $S_0 N(d_1)$, is the cost of the shares needed in the tracking portfolio. The second term represents the number of dollars borrowed at the risk-free rate. The difference in the two terms is the cost of the tracking portfolio. c_0 is the value of the option, the right-hand side is the market price of the tracking portfolio.

As the volatility of the stock price increases, the values of both put and call options on the stock increase. This is a universal property of all options. Increased volatility spreads the distribution of the future stock price, increasing both tails of the distribution.

European calls on stocks that pay no dividends are more of worth the longer the time to expiration, because:

- the final stock price is extra doubtful the longer the time to expiration
- given the same strike price, the longer the time to maturity, the lower is the present value of the strike price paid by the holder of a call option.

European puts can increase or decrease in value the longer the time to expiration. The longer the time to expiration, the more valuable are American call and put options, even for dividend-paying stocks.

The Black-Scholes call option value is increasing in the interest rate *r* because the payment of *K* for the share of stock whose call option is in the money costs less in today's dollars when the interest is higher. The opposite is true for a European & American put.

To transform the Black-Scholes formula into a more general formula that uses forward prices, replace the no-arbitrage relation, $S_0 = F_0 /(1+r_f)^T$, where F_0 is the forward price of an underlying asset in a forward contract maturing at the option expiration date, into the original Black-Scholes formula.

That formula can then be rewritten as:

$$c_0 = PV[F_0 N(d_1) - KN(d_1 - \sigma\sqrt{T})] \qquad \text{where } d_1 = \frac{\ln(F_0 / K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

and where *PV* is the risk-free discounted value of the expression inside the brackets.

This equation can be used with valuing European call options on currencies, bonds, dividend-paying stocks and commodities.

A few rules of thumb for calculating forward prices for various underlying assets:

foreign currency: multiply the present spot foreign currency rate, by the present value of a riskless unit of foreign currency paid at the forward maturity date. Multiply this value by the future value at the maturity date of a dollar paid today

riskless coupon bond: work out the current bond price less PV(coupons) and multiply this value by the future value at the maturity date of a dollar paid today

stock with dividend payments: compute the current stock price minus PV(dividends) and multiply this value by the future value at the maturity date of a dollar paid today

commodity: insert the present value of the storage costs until maturity to the current price of the commodity.

Take off the present value of the benefits(convenience yield), associated with holding an inventory of the commodity to maturity. Multiply this by the future value, at the maturity date, of a dollar paid today.

An American call option should not be prematurely exercised if the value of forward price of the underlying asset at expiration, discounted back to the present at the risk-free rate, either equals or exceeds the current price of the underlying asset. As a consequence, if one is sure that over the life of the option this will be the case, American and European options should sell for the same price if there is no arbitrage.

If the domestic interest rate is bigger than the foreign interest rate, the American option to buy domestic currency in exchange for foreign currency should sell for a similar price as the European option to do the same.

Despite a few biases in the Black-Scholes formula, it appears that the formulas work practically well when correctly implemented.