Chapter 25: Optimization

x (T) can have two specifications:

1. **Free endpoint:** means that x(T) is unrestricted and is free to be chosen optimally.

2. **Fixed endpoint:** means the final value of x(T) is specified as an equality constraint that needs to be satisfied.

General form of a **dynamic optimization problem:**

$$\text{Max} \int_0^T f[x(t), y(t)]\ dt$$

$$\text{Substitute to } \dot{x} = g[x(t), y(t)]$$

$$\text{And } x(0) = x_0 > 0 \text{ (given)}$$

In the dynamic optimization problem there are different variables:

- x(t) is referred to as the **state variable**

- y(t) is referred to as the **control variable**

- λ(t) is referred to as the **costate variable**

When we solve the dynamic optimization problem we use the **Hamiltonian function, H.**

$H(x, y, \lambda, t) = f(x, y, t) + \lambda(t)\, g(x, y, t)$

One of the following is true in dynamic maximization:

- $g(x, y, t)$ is linear on (x, y)
- $g(x, y, t)$ is concave in (x, y) and $\lambda(t) \geq 0$ for $t \in (0, T)$
- $g(x, y, t)$ is convex in (x, y) and $\lambda(t) \leq 0$ for $t \in (0, T)$

Solving the dynamic optimization problem (without discounting):

1. Determine the objective function f(x, y, t)

2. Write constraint as $\dot{x} = g(x, y, t)$

3. Construct the **Hamiltonian:**

   $H(x, y, \lambda, t) = f(x, y, t) + \lambda(t)\, g(x, y, t)$

4. Compute the FOCs of the Hamiltonian

   - dH/dy=0
   - **λ**(derivative) = - dH/dx
   - $\dot{x} = g(x, y, t)$

5. Obtain the **boundary conditions:**
   for a free endpoint:

   - $x(0) = x_0$
   - $\lambda(T) = 0$

   for a fixed endpoint:

   - $x(0) = x_0$
   - $x(T) = b$

6. Write system of differential equations in terms of x and y. This looks like:

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda \\ x \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

7. Solve the system of differential equations

8. Report x(t), y(t) and λ(t)

The boundary condition λ(T) = 0 is called a **transversality condition.**

A maximization problem with integral from 0 to T has a finite horizon. If the integral goes from 0 to ∞ then it has an infinite horizon.

We can also have and autonomous optimization problem. Such a problem involves **discounting**, with the discounting factor ρ.

General form of an **autonomous optimization problem:**

$$\text{Max} \int_0^T F[x(t), y(t)]e^{-pt}\ dt$$

Substitute to $\dot{x} = G[x(t), y(t)]$

And $x(0) = x_0$

When we want to solve an autonomous optimization problem we do not use the Hamilto but instead use the **current valued Hamiltonian, H.**

H^c (x(t), y(t), μ(t)) = F(x(t), y(t)) + μG(x(t), y(t))

The general form of an **autonomous dynamic optimization problem** with infinite horizon is:

$$\text{Max} \int_0^\infty F[x, y]e^{-pt}\ dt$$

Substitute to $\dot{x} = G[x, y]$

And $x(0) = x_0$

Solving an autonomous optimization problem (with discounting):

1. Determine the objective function F(x, y)

2. Write constraint as $\dot{x} = G(x, y'$

3. Construct the **current valued Hamiltonian:**

   $H^c (x, y, \mu) = F(x, y) + \mu(t)G(x, y)$

4. Compute the FOCs of the Hamiltonian

   - $dH^c /dy=0$
   - μ(derivative) - $\rho\mu = - dH^c /dx$
   - $\dot{x} = G(x, y)$

5. Obtain the **boundary conditions:**

   - $x(0) = x_0$
   - Plus another condition:

     - Finite horizon + free endpoint → $\mu(T) = 0$
       + fixed endpoint → $x(T) = b$
     - Infinite horizon + free endpoint → $\lim x(t) = \underline{x}$
       + fixed endpoint → $\lim x(t) = b$

6. Write as a system of differential equations in terms of x and μ.

7. Solve system of differential equations

8. Report x(t), y(t) and μ(t)